On Generalized Statistical Convergence of Double Sequences in Topological Groups

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Abstract

Following the recent introduction of the concept of $\lambda$–statistical convergence in topological groups and some inclusion relations between the sets of statistically convergent and $\lambda$–statistically convergent sequences in topological groups; we shall in this paper analogously introduce the notion of $\lambda_2$–statistical convergence of double sequences in topological groups. Some inclusion relations between the sets of statistically convergent double sequences and $\lambda_2$–statistically convergent sequences will also be proved. We shall also introduce the definition of statistically $\lambda_2$–convergence in topological groups and prove some relations.

Keywords and Phrases: $\lambda_2$–statistically Cauchy, $\lambda_2$–statistical convergence, statistically $\lambda_2$–convergent, Double sequences, topological groups. 2010 Mathematics subject classification: Primary 40F05, 40J05, 40G05

Introduction

The concept of statistical convergence was formally introduced by Fast [11] and Schoenberg [33] independently. Although statistical convergence was introduced over fifty year ago, has become an active area of research in recent years. This concepts has been applied in various areas such as summability theory by (Fridy, [13]; Salat, [31]), topological groups (Cakalli, [2], [3]), Topological spaces (Di Maio and Kocinac, [9]), locally convex spaces (Maddox, [24]), measure theory (Cheng et al., [7]; Connor and Swardson, [8]; Miller, [25]), fuzzy mathematics (Nuray and Savas, [29]; Savas, [32]). In recent years, generalization of statistical convergence has appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions, Connor and Swardson [8]. Mursaleen, [27], introduced the $\lambda$–statistical convergence for real sequences. Hazarika and Savas [21] introduced and studied $\lambda$–statistical convergence in $n$–normed spaces. Brono and Ali [1] introduced and studied the concept of $\lambda_2$–statistical convergence in $2n$–normed spaces. Hazarika and Esi [20] introduced and studied $\lambda$–statistical convergence in topological groups. In this article, in analogy to Hazarika and Esi [20] we shall introduce and study the concept $\lambda_2$–statistical convergence in topological groups as follows:

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(n,m)$ be the numbers of $(i,j)$ in $K$ such that $i \leq n$ and $j \leq m$. Then the two-dimensional analogue of natural density can be defined as follows.

The lower asymptotic density of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as...
\[ \frac{\sum_{k \in I_m} x_k}{\sum_{k \in I_m} \lambda_m} = \lim_{m \to \infty} \frac{K(n,m)}{nm} \]

In case the sequence \( \frac{K(n,m)}{nm} \) has a limit in Pringsheim’s sense then we say that \( K \) has a double natural density and is defined as

\[ \lim_{n,m} \frac{K(n,m)}{nm} = \delta_2(K). \]

For example, let \( K = \{(i^2,j^2) : i, j \in \mathbb{N}\} \). Then,

\[ \delta_2 = \lim_{n,m} \frac{K(n,m)}{nm} \leq \lim_{n,m} \frac{\sqrt{n} \sqrt{m}}{nm} = 0, \]

i.e., the set \( K \) has double natural density zero, while the set \( K = \{(i,2j) : i, j \in \mathbb{N}\} \) has double natural density \( \frac{1}{2} \).

Note that, if we set \( n = m \), we have a two dimensional natural density considered by Christopher [6].

Statistical analogue of double sequences \( x = (x_{jk}) \) was defined as follows.

**Definition 1.1**: (Mursaleen and Edely, [28]): A real double sequence \( x = (x_{jk}) \) is statistically convergent to a number \( l \) if for each \( \varepsilon > 0 \), the set

\[ \{ (j,k) : j \leq n \text{ and } k \leq m, |x_{jk} - l| \geq \varepsilon \} \]

has double natural density zero. In this case we write \( St_2 - \lim_{j,k} x_{jk} = l \) and we denote the set of all statistically convergent double sequences by \( St_2 \).

Let \( \lambda = (\lambda_m) \) be a non decreasing sequence of positive numbers tending to \( \infty \) such that

\[ \lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1. \]

The collection of such sequences \( \lambda \) will be denoted by \( \Lambda \).

The generalized de la Vallée–Poussin mean is defined by

\[ t_m(x) = \frac{1}{\lambda_m} \sum_{k \in I_m} x_k, \]

where \( I_m = [m - \lambda_m + 1, m] \).

**Definition 1.2**: (Lendler, [22]): A sequence is \( x = (x_k) \) is said to be \((V, \lambda)\)-summable to a number \( \ell \) if

\[ t_m(x) \to \ell, \quad \text{as } m \to \infty. \]

If \( \lambda_m = m \), then \((V, \lambda)\)-summability reduces to \((C,1)\)-summability. We write \( [C,1] = \left\{ x = (x_k) : \exists \ell \in \mathbb{R}, \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} |x_k - \ell| = 0 \right\} \)

and \( [V,\lambda] = \left\{ x = (x_k) : \exists \ell \in \mathbb{R}, \lim_{m \to \infty, \lambda_m} \frac{1}{\lambda_m} \sum_{k \in \lambda_m} |x_k - \ell| = 0 \right\} \).

For the sets of sequences \( x = (x_k) \) which are strongly Cesàro summable (see Freedman et al., 1978) and strongly \((V,\lambda)\)-summable to \( \ell \), i.e., \( x_k [\lambda] \to \ell \) and \( x_k [\lambda] \to \ell \) respectively.

**Definition 1.3**: (Mursaleen, [7]): A sequence \( x = (x_k) \) is said to be \( \lambda \)-statistically convergent or \( S_\lambda \)-convergent to \( \ell \) if for every \( \varepsilon > 0 \)

\[ \lim_{m \to \infty} \frac{1}{\lambda_m} |\{ k \in I_m : |x_k - \ell| \geq \varepsilon \}| = 0. \]

In this case we write

\[ S_\lambda \text{ - lim} x = \ell \text{ or } (x_k) S_\lambda \to \ell \text{ and } \]

\[ S_\lambda = \{ x = (x_k) : \exists \ell \in \mathbb{R}, S_\lambda \to \ell \} \]

It is clear that if \( \lambda_m = m \), then \( S_\lambda \) is same as \( S_1 \).

Analogously,

Let \( \lambda_2 = (\lambda_{mn}) \) be a non decreasing double sequence of positive numbers tending to \( \infty \) such that

\[ \lambda_{(m+1)(n+1)} \leq \lambda_{mn} + 1, \lambda_{11} = 1. \]

The collection of such double sequences will be denoted by \( \Delta_2 \).
The generalized de la Vallée–Poussin mean for double sequences will be defined by

\[ t_{mn}(x) = \frac{1}{\lambda_{mn}} \sum_{j, k \in I_{mn}} x_{jk}, \]

where \( I_{mn} = [mn - \lambda_{mn} + 1, mn]. \)

**Definition 1.2.1:** A double sequence \( x = (x_{jk}) \) is said to be \((V, \lambda_2)\)-summable to a number \( \ell \), if \( t_{mn} \to \ell \) as \( m, n \to \infty \).

If \( \lambda_{mn} = mn \), then \((V, \lambda_2)\)-summability reduces to \((C, 1.1)\)-summability. We write

\[ \sum_{m, n} [x_{jk} - \ell] = 0 \quad \text{and} \quad \sum_{m, n} [x_{jk} - \ell] = 0, \]

For the sets of double sequences \( x = (x_{jk}) \) which are strongly Cesàro summable (see [Mursaleen and Edely [28]]) and strongly \((V, \lambda_2)\)-summable to \( \ell \), that is \( x_{jk} \rightarrow \ell \) and \( x_{jk} \rightarrow \ell \) respectively.

**Definition 1.3.1:** A double sequence \( x = (x_{jk}) \) is said to be \( \lambda_2 \)-statistically convergent or \( \lambda_2 \)-convergent to an element \( x_{00} \) of \( X \) if for every \( \varepsilon > 0 \),

\[ \lim_{m, n \to \infty} \frac{1}{\lambda_{mn}} \sum_{j, k \in I_{mn}} |x_{jk} - \ell| = 0, \]

In this case, we write \( S_{\lambda_2} \rightarrow \ell \) or \( (x_{jk}) \rightarrow \ell \) and

\[ S_{\lambda_2} = \{ x = (x_{jk}); \exists \ell \in \mathbb{R}, S_{\lambda_2} \rightarrow x \}. \]

It is clear that if \( \lambda_{mn} = mn \), then, \( S_{\lambda_2} \) is same as \( St_2 \).

The main purpose of this article is to introduce and study the notion of \( \lambda_2 \)-statistical convergence of double sequences in topological groups and to obtain some analogue results related to statistically convergent of double sequences and \( \lambda_2 \)-statistically convergent of double sequences in topological groups.

### \( \lambda_2 \)-statistical convergence in Topological Groups

Throughout in this article \( X \) will denote a topological Hausdorff group, written additively, which satisfies the first axiom of countability. The additive identity of \( X \) will be denoted by 0. Now we introduce the definitions of \( \lambda_2 \)-statistically convergence in topological groups.

**Definition 2.1:** A double sequence \( (x_{jk}) \) of points \( X \) is said to be \( \lambda_2 \)-statistically convergent or \( \lambda_2 \)-convergent to an element \( x_{00} \) of \( X \) if for each neighbourhood \( V \) of 0,

\[ \delta_{\lambda_2}(\{ j, k \in \mathbb{N} \times \mathbb{N} : x_{jk} - x_{00} \notin V \}) = 0, \]

i.e.

\[ \lim_{m, n \to \infty} \frac{1}{\lambda_{mn}} |\{ j, k \in I_{mn} : x_{jk} - x_{00} \notin V \}| = 0. \]

In this case, we write \( S_{\lambda_2} \rightarrow x_{00} \) or \( (x_{jk}) \rightarrow x_{00} \) and we define

\[ [S_{\lambda_2}(X)] = \{ (x_{jk}) : \exists \text{ some } x_{00}, S_{\lambda_2} \rightarrow x_{00}, \forall j, k \to x_{00} \}. \]

In particular,

\[ [S_{\lambda_2}^0(X)] = \{ (x_{jk}) : S_{\lambda_2} \rightarrow x_{jk} = x_{00} \}. \]

**Definition 2.2:** A double sequence \( (x_{jk}) \) of points in \( X \) is said to be \( \lambda_2 \)-statistically Cauchy or \( \lambda_2 \)-Cauchy in \( X \) if for each neighbourhood \( V \) of 0, there is an integer \( n(V) \) such that

\[ \delta_{\lambda_2}(\{ j, k \in \mathbb{N} \times \mathbb{N} : x_{jk} - x_{nm(V)} \notin V \}) = 0. \]

For all \( j, k, m, n > n(V) \)
Throughout the article $[S_{\lambda_2}(X)][C_{\lambda_2}(X)]$ denote the set of all, $\lambda_2$ --statistically convergent double sequences and $\lambda_2$ --statistically Cauchy sequences in $X$, respectively.

**Theorem 2.1:** Every $\lambda_2$ --Statistically convergent sequence in $X$ has a unique limit.

**Proof:**

Suppose that a double sequence $x = (x_{jk})$ in $X$ has two different $\lambda_2$ --statistical limits, say $x_{00}$ and $y_{00}$. Since $X$ is a Hausdorff space there exists a neighbourhood $V$ of 0 such that $x_{00} - y_{00} \notin V$. Also for each neighbourhood $V$ of zero, there exists a symmetric neighbourhood $W$ of 0 such that $W + W \subseteq V$. Write $z_{jk} = x_{00} - y_{00}$ for all $j, k \in \mathbb{N}$. Therefore, for all $m, n \in \mathbb{N}$,

$$\{jk \in l_m: z_{jk} \notin V\} \subseteq \{jk \in l_m: x_{00} - x_{jk} \notin W\} \cup \{jk \in l_m: x_{jk} - y_{00} \notin W\}.$$  

Now it follows from this inclusion that, for all $m, n \in \mathbb{N}$,

$$\frac{1}{\lambda_{mn}} \left| \{jk \in l_m: z_{jk} \notin V\} \right| \leq \frac{1}{\lambda_{mn}} \left| \{jk \in l_m: x_{00} - x_{jk} \notin W\} \right| + \frac{1}{\lambda_{mn}} \left| \{jk \in l_m: x_{jk} - y_{00} \notin W\} \right|.$$  

Since $S_{\lambda_2} - \lim x_{jk} = x_{00}$ and $S_{\lambda_2} - \lim x_{jk} = y_{00}$, we obtain

$$\frac{1}{\lambda_{mn}} \left| \{jk \in l_m: x_{jk} \notin W\} \right| \leq \frac{1}{\lambda_{mn}} \left| \{jk \in l_m: x_{00} - x_{jk} \notin W\} \right| + \frac{1}{\lambda_{mn}} \left| \{jk \in l_m: x_{jk} - y_{00} \notin W\} \right|.$$  

Therefore, $1 \leq 0 + 0 = 0$. Which is a, contradiction, hence $x_{00} = y_{00}$.

**Theorem 2.2:** A sequence $(x_{jk})$ is $S_{\lambda_2}$ --convergent to $x_{00}$ in $X$ if and only if for each neighbourhood $V$ of 0, there exists a subsequence $x_{j'k'(m,n)}$ of $(x_{jk})$ such that $\lim_{m,n} x_{j'k'(m,n)} = x_{00}$ and

$$\delta_{\lambda_2}\left(\{j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{j'k'(m,n)} \notin V\}\right) = 0.$$  

**Proof:** Let $(x_{jk})$ be a double sequence in $X$ such that $S_{\lambda_2} - \lim_{jk} x_{jk} = x_{00}$. Let $\{V_i\}$ be a nested base of neighbourhoods of 0. We write $E^{(i)} = \{j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{00} \notin V_i\}$ for any positive integer $i$. Then for each $i$, we have $E^{(i+1)} \subseteq E^{(i)}$ and

$$\lim_{\lambda_{mn}} \frac{|E^{(i)} \cap l_{mn}|}{\lambda_{mn}} = 1.$$  

Choose $i(1)$ such that $m > n(1)$ then $|E^{(1)} \cap l_{mn}| > 0$ i.e., $E^{(1)} \cap l_{mn} \neq \emptyset$. Then for each positive integer $m$ such that $i(1) \leq l < i(2)$, choose $j'k'(m,n) \in l_{mn}$ such that $j'k'(l) \in l_{mn}$ i.e., $x_{j'k'(m,n)} - x_{00} \in V_1$. In general, choose $i(p + 1) > i(p)$ such that $l > i(p + 1)$, then $E^{(p+1)} \cap l_{mn} \neq \emptyset$. Then for all $l$ satisfying $i(p) \leq l < i(p + 1)$, choose $j'k'(p) \in l_{mn}$ i.e., $x_{j'k'(m,n)} - x_{00} \in V_p$. Hence it follows that $\lim_{m,n} x_{j'k'(m,n)} = x_{00}$ let $V$ be any neighbourhood of 0. Then there is a symmetric neighbourhood $W$ of 0 such that $W + W \subseteq V$. Now we have

$$\{j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{j'k'(m,n)} \notin V\} \subseteq \{j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{00} \notin W\} \cup \{m, n \in \mathbb{N} \times \mathbb{N}: x_{j'k'(m,n)} - x_{00} \notin W\}.$$  

Since $S_{\lambda_2} - \lim_{jk} x_{jk} = x_{00}$ and $\lim_{m,n} x_{j'k'(m,n)} = x_{00}$, this implies that

$$\delta_{\lambda_2}\left(\{j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{j'k'(m,n)} \notin V\}\right) = 0.$$
Conversely, suppose for each neighbourhood $V$ of $0$ there exists a subsequence $(x_{jk(m,n)})$ of 
$(x_{jk})$ such that $\lim_{m,n} x_{j'k'(m,n)} = x_{00}$ and

$\delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} = x_{j'k'}(m,n) \notin V\}) = 0$. 

Since $V$ is a neighbourhood of $0$, we may choose a symmetric neighbourhood $W$ of $0$ such that $W + W \subset V$. Then we have

$\delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{00} \notin V\}) \leq \delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{j'k'}(m,n) \notin W\}) + \delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: x_{j'k'}(m,n) - x_{00} \notin W\})$.

Therefore

$\delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{00} \notin V\}) = 0$.

This completes the proof.

**Theorem 2.3:** If $\lim_{j,k} x_{jk} = x_{00}$ and $S_{x_{jk}} - \lim_{j,k} y_{jk} = 0$, then $S_{x_{jk}} - \lim_{j,k} (x_{jk} + y_{jk}) = x_{00}$.

**Proof:** Let $V$ be any neighbourhood of $0$. Then we may choose a symmetric neighbourhood $W$ of $0$ such that $W + W \subset V$. Since $\lim_{j,k} x_{jk} = x_{00}$, then there exist integers $j \geq m_0$ and $k \geq n_0$ implies that $x_{jk} - x_{00} \in W$. Hence

$\delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{00} \notin W\}) = 0$.

By assumption $S_{x_{jk}} - \lim_{j,k} y_{jk} = 0$, then we have $\delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{00} \notin W\}) = 0$.

Thus

$\delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: (x_{jk} - x_{00} + y_{jk}) \in V\}) \leq \delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{00} \notin W\}) + \delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: y_{jk} \notin W\})$.

This implies that

$S_{x_{jk}} - \lim_{j,k} (x_{jk} + y_{jk}) = \lim_{j,k} x_{jk}$.

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**Theorem 2.4:** If a double sequence $(y_{jk})$ is $S_{x_{jk}}$ -convergent to $y_{00}$ in $X$, then there are double sequences $(y_{jk})$ and $(z_{jk})$ such that $x_{jk} = y_{jk} + z_{jk}$, for each $j, k \in \mathbb{N}$, $S_{x_{jk}} - \lim_{j,k} y_{jk} = x_{00}$, $\delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} = y_{jk}\}) = 0$ and $(z_{jk})$ is $S_{x_{jk}}$-null sequence.

**Proof:** Let $\{V_i\}$ be a nested base neighbourhood of $0$. Take $n_0 = 0$ and choose and increasing sequence $(n_i)$ of positive integers such that $\delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{00} \notin V_i\}) < \frac{1}{i}$ for $j, k > n_i$.

Let us define the double sequences $(y_{jk})$ and $(z_{jk})$ as follows:

- $y_{jk} = x_{jk}$ and $z_{jk} = 0$, if $0 < j, k \leq n_1$,
- $y_{jk} = x_{jk}$ and $z_{jk} = 0$, if $x_{jk} - x_{00} \in V_i$,
- $y_{jk} = x_{jk}$ and $z_{jk} = x_{jk} - x_{00}$, if $x_{jk} - x_{00} \notin V_i$.

We have to show that (i) $\lim_{j,k} y_{jk} = x_{00}$ and (ii) $(z_{jk})$ is a $S_{x_{jk}}$-null sequence.

(i) Let $V$ be any neighbourhood of $0$. We must choose a positive integer $i$ such that $V_i \subset V$. Then $y_{jk} - x_{00} = x_{jk} - x_{00} \notin V_i$ for $j, k > n_i$. If $x_{jk} - x_{00} \in V_i$, then $y_{jk} - x_{00} = x_{00} - x_{00} = 0 \in V$. Hence $\lim_{j,k} y_{jk} = x_{00}$.

(ii) It is enough to show that $\delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: z_{jk} \neq 0\}) = 0$. For any neighbourhood $V$ of $0$, we have $\delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: z_{jk} \notin V\}) \leq \delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: z_{jk} \neq 0\})$. If $n_p < j, k \leq n_{p+1}$, then $\delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: z_{jk} \neq 0\}) \subset \{j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{00} \notin V'_p\}$. If $p > i$ and $n_p < j, k \leq n_{p+1}$, then $\delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: z_{jk} \neq 0\}) \leq \delta_{x_{jk}}(\{(j, k \in \mathbb{N} \times \mathbb{N}: x_{jk} - x_{00} \notin V'_p\}) < \frac{1}{p} < \frac{1}{i} < \varepsilon$. 

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This implies that $\delta_{\lambda_2}(\{(j, k) \in \mathbb{N} \times \mathbb{N} : z_{jk} \neq 0\}) = 0$. Hence $(z_{jk})$ is an $S_{\lambda_2}$-null sequence.

**Theorem 2.5:** Let $x = (x_{jk})$ be a double sequence in $X$. If there is a $S_{\lambda_2}$-convergent sequence $y = (y_{jk})$ in $X$ such that $\delta_{\lambda_2}(\{(j, k) \in \mathbb{N} \times \mathbb{N} : y_{jk} \neq x_{jk} \in V\}) = 0$ then $x_{jk}$ is also $S_{\lambda_2}$-convergent.

**Proof:** Suppose that $\delta_{\lambda_2}(\{(j, k) \in \mathbb{N} \times \mathbb{N} : y_{jk} \neq x_{jk} \in V\}) = 0$ and $S_{\lambda_2} \lim y_{jk} = x_{00}$. Then for every neighbourhood $V$ of 0, we have $\delta_{\lambda_2}(\{(j, k) \in \mathbb{N} \times \mathbb{N} : y_{jk} \neq x_{00} \in V\}) = 0$.

Now

$$\{j, k \in \mathbb{N} \times \mathbb{N} : x_{jk} \neq x_{00} \in V\} \subseteq \{j, k \in \mathbb{N} \times \mathbb{N} : y_{jk} \neq x_{jk} \in V\} \cup \{j, k \in \mathbb{N} \times \mathbb{N} : y_{jk} \neq x_{00} \in V\}$$

$$\Rightarrow \delta_{\lambda_2}(\{(j, k) \in \mathbb{N} \times \mathbb{N} : x_{jk} \neq x_{00} \in V\}) \leq \delta_{\lambda_2}(\{(j, k) \in \mathbb{N} \times \mathbb{N} : y_{jk} \neq x_{jk} \in V\}) + \delta_{\lambda_2}(\{(j, k) \in \mathbb{N} \times \mathbb{N} : y_{jk} \neq x_{00} \in V\}).$$

Therefore, we have $\delta_{\lambda_2}$

This completes the proof.

**Theorem 2.6:** Let $x = (x_{jk})$ be a double sequence in $X$. Then $(x_{jk}) \rightarrow x_{00}$ implies $(x_{jk}) \lim \inf_{mn} \frac{\lambda_{mn}}{mn} > 0$.

**Proof:** Suppose first that $\liminf_{mn} \frac{\lambda_{mn}}{mn} > 0$ and $(x_{jk}) \rightarrow x_{00}$. Let $V$ be any neighbourhood of 0. Then for all $m, n \in \mathbb{N}$,

$$\frac{1}{mn} \{j \leq m, k \leq n : x_{jk} \neq x_{00} \in V\} \geq \frac{1}{mn} \{j \in l_{mn} : x_{jk} \neq x_{00} \in V\} \geq \frac{\lambda_{mn}}{mn} \frac{1}{mn} \{j \in l_{mn} : x_{jk} \neq x_{00} \in V\}.$$

Since $(x_{jk}) \rightarrow x_{00}$. Therefore this inequality implies that $(x_{jk}) \rightarrow x_{00}$, i.e. $[l_{t2}] \subset [S_{\lambda_2}]$.

**Theorem 2.7:** Let $x = (x_{jk})$ be a double sequence in $X$. Then $S_{t2} = S_{\lambda_2}$ if $\lim_{mn} \frac{\lambda_{mn}}{mn} = 1$.

**Proof:** Suppose that $\lim_{mn} \frac{\lambda_{mn}}{mn} = 1$. Let $V$ be any neighbourhood of 0. We observe that

$$\frac{1}{mn} \{j \leq m, k \leq n : x_{jk} \neq x_{00} \in V\} \leq \frac{1}{mn} \{j \in l_{mn} : x_{jk} \neq x_{00} \in V\} + \frac{1}{mn} \{j \in l_{mn} : x_{00} \in V\} \leq \frac{1}{mn} \{j \in l_{mn} : x_{jk} \neq x_{00} \in V\} + \frac{1}{mn} \frac{\lambda_{mn}}{mn} \frac{1}{mn} \{j \in l_{mn} : x_{jk} \neq x_{00} \in V\}.$$

This implies that $x$ is statistically convergent to $x_0$ in $X$ if $x$ is $\lambda_2$-statistically convergent to $x_0$.

Thus $[S_{\lambda_2}] \subset [S_{t2}]$.

Since $\lim_{mn} \frac{\lambda_{mn}}{mn} = 1$ implies that $\liminf_{mn} \frac{\lambda_{mn}}{mn} > 0$, then from Theorem 5, we have $S_{t2} \subset S_{\lambda_2}$.

Hence $[S_{t2}] = [S_{\lambda_2}]$.

1. **Statistical $\lambda_2$-convergence in Topological groups**

**Definition 3.1:** A double sequence $(x_{jk})$ of points in $X$ is said to be statistically $\lambda_2$-convergence or $S_{\lambda_2}$-convergent to an element of $X$ if for each neighbourhood $V$ of 0, the set $K(\lambda_2) = \{n, m \in \mathbb{N} : t_{mn}(x) - x_{00} \in V\}$ has
natural density zero, or equivalently, 
$$\delta(K(\lambda_2)) = 0, \text{ i.e.}$$

$$\lim_{mn} \frac{1}{mn} \left| \left\{ n, m \in \mathbb{N} : t_{mn}(x) - x_{00} \notin V \right\} \right| = 0.$$ 

In this case we write $S_{\delta \lambda_2} - \lim_{jk} x_{jk} = S_{\delta \lambda_2} - \lim_{jk} x_{jk} = x_{00}$ or $(x_{jk}) \rightarrow x_{00}$ and we define

$$\left[ S_{\delta \lambda_2}(x) \right] = \left\{ (x_{jk}) : \text{for some } x_{00}, \ S_{\delta \lambda_2} - \lim_{jk} x_{jk} = x_{00} \right\}.$$ 

**Theorem 3.1:** A double sequence $x = (x_{jk})$ in $X$ is statistically $\lambda_2$–convergent to $x_{00}$ if and only if there exists a $K = \{j_1, k_1 < j_2, k_2 < j_3, k_3 < \cdots < j_m, k_n < \cdots \} \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta(K) = 1$ and $\lambda_2 - \lim x_{jmk_n} = x_{00}.$

**Proof:** Suppose that there exists a set $K = \{j_1, k_1 < j_2, k_2 < j_3, k_3 < \cdots < j_m, k_n < \cdots \} \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta(K) = 1$ and $\lambda_2 - \lim x_{jmk_n} = x_{00}.$ For each neighbourhood $V$ of $0$, there exists a positive integer $N$ such that $(t_{nm}(x) - x_{00}) \in V$ for $m, n > N$.

We put $K(\lambda_2) = \{m, n \in \mathbb{N} : t_{mn}(x) - x_{00} \notin V \}$ and $K_1 = \{j_1, k_1 < j_2, k_2 < j_3, k_3 < \cdots < j_m, k_n < \cdots \} \subseteq \mathbb{N} \times \mathbb{N}$ which implies that $\delta(K(\lambda_2)) = 0$. Hence $x = (x_{jk})$ is statistically $\lambda_2$–convergent to $x_{00}$.

Conversely, suppose that $x = (x_{jk})$ is statistically $\lambda_2$–convergent to $x_{00}$. Let $\{V_r\}$ be a nested base neighbourhoods of $0$. We put

$$K_r(\lambda_2) = \{m, n \in \mathbb{N} : t_{jmk_n}(x_{jk}) - x_{00} \notin V_r \}$$

and $M_r(\lambda_2) = \{m, n \in \mathbb{N} : t_{jmk_n}(x_{jk}) - x_{00} \notin V_r \}.$

Then $\delta(K_r(\lambda_2)) = 0$.

(1) $M_r(\lambda_2) \supset M_{r+1}(\lambda_2) \supset \cdots$

(2) $\delta(M_r(\lambda_2)) = 1, r \in \mathbb{N}.$

We show that for $m, n \in M_r(\lambda_2), \lambda_2 - \lim x_{jmk_n} = x_{00}$. Suppose that $(x_{jmk_n})$ is not $\lambda_2$–convergent to $x_0$. For each neighbourhood $V$ of $0$, there exists a positive integer $N$ such that $(t_{nm}(x) - x_{00}) \notin V$ for $m > N$.

Let $M(\lambda_2) = \{m, n \in \mathbb{N} : t_{jmk_n}(x_{jk}) - x_{00} \notin V \}$ and $V \supset V_r$ for $r \in \mathbb{N}$. Then $\delta(M(\lambda_2)) = 0$. By (1), we have $M_r(\lambda_2) \subset M(\lambda_2)$ and hence $\delta(M_r(\lambda_2)) = 0$, which contradict (2). Therefore $(x_{jmk_n})$ is $\lambda_2$–convergent to $x_{00}$. This completes the proof.

**Theorem 3.2:** A double sequence $x = (x_{jk})$ in $X$ is $\lambda_2$–statistically convergent to $x_{00}$ if and only if there exists a set $K = \{j_1, k_1 < j_2, k_2 < j_3, k_3 < \cdots < j_m, k_n < \cdots \} \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta(K) = 1$ and $\lambda_2 - \lim x_{jmk_n} = x_{00}.$

**Proof:** Proof of the theorem follows from theorem 2.2 and theorem 3.1

**Conclusion:** This paper has successfully extended the concept on generalized statistical convergence in topological groups to double sequences using analogy. However, the concept can be explored for possible further extension to triple sequences or into topological semigroups.

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RE: CONFLICT OF INTEREST

On the above subject matter, we think there is some mixed up; As we have not at any time submitted the name Biphan Hazarika as part of the authorship of the manuscript under consideration. You may wish to note that, a paper titled “On generalized statistical convergence in topological groups” published by B. Hazarika and A. Esi in Rend. Sem. Mat. Univ. Politec. Torino, 70(2012), 497-505. Has been cited being the foundation of this manuscript submitted to your esteemed journal.

Finally, you may also wish to note that we have not earlier submitted this manuscript to any other journals.

Thank You

Yours faithfully,

Signed

A. M. Brono
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